

Generalized Symmetric Divergence Measures and Metric Spaces

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Abstract

Recently, Taneja [7] studied two one parameter generalizations of *J-divergence*, *Jensen-Shannon divergence* and *Arithmetic-Geometric divergence*. These two generalizations in particular contain measures like: *Hellinger discrimination*, *symmetric chi-square divergence*, and *triangular discrimination*. These measures are well known in the literature of Statistics and Information theory. In this paper our aim is to prove metric space properties for square root of these two symmetric generalized divergence measures.

Key words: *J-divergence; Jensen-Shannon divergence; Arithmetic-Geometric divergence; Metric Space.*

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1 Introduction

Let

$$\Gamma_n = \left\{ P = (p_1, p_2, \dots, p_n) \left| p_i > 0, \sum_{i=1}^n p_i = 1 \right. \right\}, \quad n \geq 2,$$

be the set of all complete finite discrete probability distributions. For all $P, Q \in \Gamma_n$, let us consider two generalized symmetric divergence measures. These measures are well known in the literature on information theory and statistics.

Let us consider the measure

$$\xi_s(P||Q) = \begin{cases} L_s(P||Q) = [s(s-1)]^{-1} \left[\sum_{i=1}^n \left(\frac{p_i^s + q_i^s}{2} \right) \left(\frac{p_i + q_i}{2} \right)^{1-s} - 1 \right], & s \neq 0, 1 \\ I(P||Q) = \frac{1}{2} \left[\sum_{i=1}^n p_i \ln \left(\frac{2p_i}{p_i + q_i} \right) + \sum_{i=1}^n q_i \ln \left(\frac{2q_i}{p_i + q_i} \right) \right], & s = 1 \\ T(P||Q) = \sum_{i=1}^n \left(\frac{p_i + q_i}{2} \right) \ln \left(\frac{p_i + q_i}{2\sqrt{p_i q_i}} \right), & s = 0 \end{cases} \quad (1.1)$$

for all $P, Q \in \Gamma_n$

The measure (1.1) was studied for the first time by Taneja [6] and is called *generalized symmetric arithmetic and geometric mean divergence*. The measure (1.1) admits the following particular cases:

- (i) $\xi_{-1}(P||Q) = \frac{1}{4}\Delta(P||Q)$.
- (ii) $\xi_1(P||Q) = I(P||Q)$.
- (iii) $\xi_{1/2}(P||Q) = 4 d(P||Q)$.
- (iv) $\xi_0(P||Q) = T(P||Q)$.
- (v) $\xi_2(P||Q) = \frac{1}{16}\Psi(P||Q)$.

where

$$\Delta(P||Q) = \sum_{i=1}^n \frac{(p_i - q_i)^2}{p_i + q_i},$$

and

$$d(P||Q) = 1 - \sum_{i=1}^n \left(\frac{\sqrt{p_i} + \sqrt{q_i}}{2} \right) \left(\sqrt{\frac{p_i + q_i}{2}} \right).$$

are the *triangular discrimination* and *d-divergence* respectively. The measures $I(P||Q)$ and $T(P||Q)$ are the well-known Jensen-Shannon divergence [5, 1] and Arithmetic-Geometric mean divergence [6], respectively.

Let us consider now the other measure

$$\zeta_s(P||Q) = \begin{cases} J_s(P||Q) = [s(s-1)]^{-1} \left[\sum_{i=1}^n (p_i^s q_i^{1-s} + p_i^{1-s} q_i^s) - 2 \right], & s \neq 0, 1 \\ J(P||Q) = \sum_{i=1}^n (p_i - q_i) \ln \left(\frac{p_i}{q_i} \right), & s = 0, 1 \end{cases} \quad (1.2)$$

for all $P, Q \in \Gamma_n$

The measure (1.2) can be seen in Burbea and Rao [1] and Taneja [7]. The expression (1.2) admits the following particular cases:

- (i) $\zeta_{-1}(P||Q) = \zeta_2(P||Q) = \frac{1}{2}\Psi(P||Q)$,
- (ii) $\zeta_0(P||Q) = \zeta_1(P||Q) = J(P||Q)$,
- (iii) $\zeta_{1/2}(P||Q) = 8 h(P||Q)$,

where

$$\Psi(P||Q) = \chi^2(P||Q) + \chi^2(Q||P) = \sum_{i=1}^n \frac{(p_i - q_i)^2 (p_i + q_i)}{p_i q_i},$$

and

$$h(P||Q) = 1 - B(P||Q) = \frac{1}{2} \sum_{i=1}^n (\sqrt{p_i} - \sqrt{q_i})^2, \quad (1.3)$$

are the *symmetric χ^2 -divergence* and *Hellinger's discrimination* respectively. The measure $J(P||Q)$ is the well-known J-divergence [4]. For detailed study of the measures (1.1) and (1.2) refer to Taneja [6, 7].

The symmetric divergence measures (1.1) and (1.2) admit several particular cases. An inequality among these measures [6] is given by

$$\frac{1}{4}\Delta(P||Q) \leq I(P||Q) \leq h(P||Q) \leq 4d(P||Q) \leq \frac{1}{8}J(P||Q) \leq T(P||Q) \leq \frac{1}{16}\Psi(P||Q). \quad (1.4)$$

An improvement over the inequalities given in (1.4) can be seen in Taneja [8, 9].

In this paper our aim is to prove metric space properties of the square root of the measures (1.1) and (1.2).

2 Generalized Divergence Measures and Metric Spaces

In this section we shall prove the metric space property of the square root of the measures given in (1.1) and (1.2).

2.1 JS and AG – Divergences of Type s

Let the function $\zeta_s(p, q) : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be defined as

$$\zeta_s(p, q) = \begin{cases} L_s(p, q) = \frac{1}{s(s-1)} \left[\left(\frac{p^s+q^s}{2} \right) \left(\frac{p+q}{2} \right)^{1-s} - \left(\frac{p+q}{2} \right) \right], & s \neq 0, 1 \\ I(p, q) = \frac{p}{2} \ln \left(\frac{2p}{p+q} \right) + \frac{q}{2} \ln \left(\frac{2q}{p+q} \right), & s = 0 \\ T(p, q) = \left(\frac{p+q}{2} \right) \ln \left(\frac{p+q}{2\sqrt{pq}} \right), & s = 1 \end{cases} \quad (2.1)$$

In view of (2.1), we can write

$$\zeta_s(P||Q) = \sum_{i=1}^n \zeta_s(p_i, q_i), \quad (2.2)$$

for all $P, Q \in \Gamma_n$

Theorem 2.1. *The measure given by $\sqrt{\zeta_s(p, q)}$ is a metric over \mathbb{R}^+ .*

Proof. (i) Initially we shall prove the result for $s \neq 0, 1$. It is sufficient to show the triangle inequality:

$$\sqrt{L_s(p, q)} \leq \sqrt{L_s(p, r)} + \sqrt{L_s(r, q)}, \quad \forall p, q, r \in \mathbb{R}^+, \quad s \neq 0, 1 \quad (2.3)$$

Let us write

$$K_{pq}(r) = \sqrt{L_s(p, r)} + \sqrt{L_s(r, q)}.$$

Now we shall prove that K_{pq} has only one minimum at $r = p = q$. The derivative of K_{pq} with respect to r is

$$K'_{pq}(r) = \frac{L'_s(p, r)}{2\sqrt{L_s(p, r)}} + \frac{L'_s(r, q)}{2\sqrt{L_s(r, q)}},$$

where

$$\begin{aligned} L'_s(p, r) &= \frac{d}{dr} L_s(p, r) \\ &= \frac{(1-s)r^{-s}(p+r)^s + (p^{1-s} + r^{1-s})s(p+r)^{s-1} - 2^s}{s(s-1)2^{s+1}} \\ &\stackrel{\frac{p}{r}=t}{=} \frac{(1-s)(1+t)^s + s(t^{1-s} + 1)(t+1)^{s-1} - 2^s}{s(s-1)2^{s+1}}. \end{aligned}$$

Also, we can write

$$\sqrt{L_s(p, r)} \stackrel{\frac{p}{r}=t}{=} \sqrt{r} \sqrt{L_s(t, 1)},$$

Multiply K'_{pq} by $2\sqrt{r}$ and define the function $h(t)$ by setting

$$\frac{2\sqrt{r}L'_s(p, r)}{\sqrt{L_s(p, r)}} \stackrel{\frac{p}{r}=t}{=} h_{L_s}(t) = \frac{n_{L_s}(t)}{d_{L_s}(t)},$$

where

$$n_{L_s}(t) = \left. \frac{d}{dr} L_s(p, r) \right|_{\frac{p}{r}=t}$$

and

$$d_{L_s}(t) = \sqrt{L_s(t, 1)}.$$

Thus the sign of $h_{L_s}(t)$ depends only on the sign of $n_{L_s}(t)$. We have

$$n'_{L_s}(t) = -\frac{(1+t)^{s-2}}{2^{s+1}} [(1+t)(1+t^{-s}) + (1+t^{1-s})].$$

This give $n'_{L_s}(t) < 0$, $\forall t > 0$ and $\forall s \Rightarrow n_{L_s}(t)$ decreases monotonically in $(0, +\infty)$.

As $h_{L_s}(1) = 0$, $n_{L_s}(t)$ changes the sign at $t = 1$, therefore $h_{L_s}(t)$ changes the sign at $t = 1$. This gives

$$h_{L_s}(t) \begin{cases} > 0, & t < 1 \\ < 0, & t > 1 \end{cases}$$

for any s .

As $\frac{p}{r} = t$, then $\frac{q}{r} = \frac{q}{p} \frac{p}{r} = \beta t$, where $\beta = \frac{q}{p}$. Therefore,

$$2\sqrt{r} \frac{dK_{L_s}}{dr} = h_{L_s}(t) + h_{L_s}(t\beta).$$

Now, suppose $\beta > 1$ ($q > p$), this give:

- for $t < \frac{1}{\beta}$: $h_{L_s}(t)$ and $h_{L_s}(\beta t)$ have the same sign +
- for $t > 1$: $h_{L_s}(t)$ and $h_{L_s}(\beta t)$ have the same sign -
- for $t \in \left(\frac{1}{\beta}, 1\right) \Rightarrow t\beta > 1 \Rightarrow h_{L_s}(t\beta) < 0$
- for $t \in \left(\frac{1}{\beta}, 1\right) \Rightarrow h_{L_s}(t) > 0$

Finally, we have for $t \in \left(\frac{1}{\beta}, 1\right)$, $h_{L_s}(t) > 0$ e $h_{L_s}(t\beta) < 0$. Since $|h_{L_s}(t)| > |h_{L_s}(t\beta)|$ (h_{L_s} is monotonically decreasing), then $h_{L_s}(t) + h_{L_s}(\beta t) > 0$. For $t > 1$, $h_{L_s}(t) < 0$ e $h_{L_s}(t\beta) < 0$ and $h_{L_s}(t) + h_{L_s}(\beta t) < 0$.

Therefore, $\frac{dK_{L_s}}{dr}$ indeed changes from positive to negative sign at $t = 1$ ($r = p$) so that there is a minimum at $t = 1$. Now, we shall show that this happens only once.

Since h_{L_s} is monotonically decreasing this implies that $h'_{L_s} < 0$ and we know that the function h_{L_s} changes the sign only once. This gives

$$\frac{d}{dt} (h_{L_s}(t) + h_{L_s}(t\beta)) = h'_{L_s}(t) + h'_{L_s}(t\beta) < 0,$$

The case $q < p$ can be investigated in a similar fashion. Symmetry of L_s allows us to take $t = \frac{q}{r}$ and $\frac{p}{r} = \beta t$ with $\beta = \frac{p}{q} > 1$. From this we conclude that there is a minimum at $r = q$.

Repeating the same process by substituting $t := \frac{q}{r}$ and $\frac{p}{r} = \beta t$ with $\beta = \frac{p}{q}$ we conclude that the function K'_{pq} also changes the sign at $r = q$. This proves the result (2.3) for $s \neq 0, 1$.

(ii) Now we shall prove the result for $s = 1$. We have to show that

$$\sqrt{T(p, q)} \leq \sqrt{T(p, r)} + \sqrt{T(r, q)}, \quad \forall p, q, r \in \mathbb{R}^+.$$

Let us write

$$T_{pq}(r) = \sqrt{T(p, r)} + \sqrt{T(r, q)},$$

then obviously,

$$T'_{pq}(r) = \frac{T'(p, r)}{2\sqrt{T(p, r)}} + \frac{T'(r, q)}{2\sqrt{T(r, q)}}.$$

Now, we have

$$\begin{aligned}
T'(p, r) &= \frac{d}{dr} T(p, r) \\
&= \frac{1}{2} \ln \left(\frac{p+r}{2\sqrt{pr}} \right) + \left(\frac{p+r}{2} \right) \frac{d}{dr} \left[\ln \left(\frac{p+r}{2\sqrt{pr}} \right) \right] \\
&= \frac{1}{2} \ln \left(\frac{p+r}{2\sqrt{pr}} \right) + \frac{r-p}{4r}.
\end{aligned}$$

This give

$$\frac{T'(p, r)}{2\sqrt{T(p, r)}} = \frac{h_T(t)}{\sqrt{32r}} \Big|_{t=\frac{p}{r}}$$

where

$$h_T(t) := \frac{2 \ln \left(\frac{t+1}{2\sqrt{t}} \right) + (1-t)}{\sqrt{(t+1) \ln \left(\frac{(1+t)}{2\sqrt{t}} \right)}}.$$

Let us take now $\frac{q}{r} = \beta t$ where $\beta = \frac{q}{p}$, then we can write

$$\sqrt{32r} T'_{pq}(r) \Big|_{t=\frac{p}{r}} = h_T(t) + h_T(\beta t).$$

Let us study now the function $h_T(t)$. Call $n_T(x)$ and $d_T(t)$ the functions in the numerator and denominator of $h_T(t)$, respectively. Since we know that $d_T(t) > 0$, $\forall x > 0$, then the sign of $h_T(t)$ is determined by $n_T(t)$.

Now,

$$n'_T(t) = 2 \frac{\sqrt{t}}{(t+1)} \frac{d}{dt} \left(\frac{t+1}{\sqrt{t}} \right) - 1 = -\frac{(1+t^2)}{t(t+1)}.$$

From this we conclude that $n_T(t)$ is decreasing $\forall t > 0$ and $n'_T(t) \neq 0$, $\forall t \in \mathbb{R}$. Since $n_T(1) = 0$ then $n_T(t)$ changes the sign at $t = 1$. This gives that $h_T(t)$ changes the sign at $t = 1$.

Thus we conclude that $h_T(t)$ is decreasing function of x and

$$h_T(t) \begin{cases} > 0, & t < 1 \\ < 0, & t > 1 \end{cases}$$

Set $\beta > 1$. In this case,

- for $t < \frac{1}{\beta}$: $h_T(t)$ and $h_T(\beta t)$ have the same sign +
- for $t > 1$: $h_T(t)$ and $h_T(\beta t)$ have the same sign -

- for $t \in \left(\frac{1}{\beta}, 1\right) \Rightarrow t\beta > 1 \Rightarrow h_T(t\beta) < 0$
- for $t \in \left(\frac{1}{\beta}, 1\right) \Rightarrow h_T(t) > 0$

Finally, for $t \in \left(\frac{1}{\beta}, 1\right)$, $h_T(t) > 0$ e $h_T(t\beta) < 0$. Thus we observe that the sign of $h_T(t) + h_T(\beta t)$ may change in $\left(\frac{1}{\beta}, 1\right)$. Now, we shall show that this happens only once.

Since h'_T is monotonically decreasing this implies that $h'_T < 0$ and we know that the function h_T changes the sign only once. This gives

$$\frac{d}{dt} (h_T(t) + h_T(t\beta)) = h'_T(t) + h'_T(t\beta) < 0,$$

Repeating the same process by substituting $t := \frac{q}{r}$ and $\frac{p}{r} = \beta t$ with $\beta = \frac{p}{q}$ we conclude that the function T'_{pq} also changes the sign at $r = q$.

(iii) For $s = 0$ the result is already given in Endres and Schindelin [3]. \square

2.2 J – Divergences of Type s

Let the function $\xi_s(p, q) : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be defined as

$$\xi_s(p, q) = \begin{cases} J_s(p, q) = [s(s-1)]^{-1} [p^s q^{1-s} + p^{1-s} q^s - (p+q)], & s \neq 0, 1 \\ J(p, q) = (p-q) \ln\left(\frac{p}{q}\right) & s = 0, 1 \end{cases} \quad (2.4)$$

In view of (2.4), we can write

$$\xi_s(P||Q) = \sum_{i=1}^n \xi_s(p_i, q_i), \quad (2.5)$$

for all $P, Q \in \Gamma_n$

Theorem 2.2. *The measure given by $\sqrt{\xi_s(p, q)}$ is a metric space over \mathbb{R}^+ .*

Proof. (i) Initially we shall prove the result for $s \neq 0, 1$. It is sufficient to show the triangle inequality:

$$\sqrt{J_s(p, q)} \leq \sqrt{J_s(p, r)} + \sqrt{J_s(r, q)}, \quad \forall p, q, r \in \mathbb{R}^+. \quad (2.6)$$

Let us write

$$F_{pq}(r) = \sqrt{J_s(p, r)} + \sqrt{J_s(r, q)}, \quad p \neq q,$$

then obviously,

$$F'_{pq}(r) = \frac{J'_s(p, r)}{2\sqrt{J_s(p, r)}} + \frac{J'_s(r, q)}{2\sqrt{J_s(r, q)}}.$$

Now, we have

$$\begin{aligned}
J'_s(p, r) &= \frac{d}{dr} J_s(p, r) \\
&= \frac{(1-s)p^s r^{-s} + s p^{1-s} r^{s-1} - 1}{s(s-1)} \\
&\stackrel{\frac{p}{r}=t}{=} \frac{t^s r + t^{1-s} r - tr - r}{s(s-1)}
\end{aligned}$$

Also, we can write

$$\sqrt{J_s(p, r)} \stackrel{p=rt}{=} \sqrt{r} \sqrt{J_s(t, 1)},$$

Let us write

$$\sqrt{r} \frac{J'_s(p, r)}{\sqrt{J_s(t, 1)}} \stackrel{\frac{p}{r}=t}{=} h_{J_s}(t) = \frac{n_{J_s}(t)}{d_{J_s}(t)},$$

where

$$n_{J_s}(t) = \left. \frac{d}{dr} J_s(p, r) \right|_{\frac{p}{r}=t}$$

and

$$d_{J_s}(t) = \sqrt{J_s(t, 1)}.$$

Thus the sign of $h_{J_s}(t)$ depends on the sign of $n_{J_s}(t)$.

$$n'_{J_s}(t) = -t^{s-1} - t^{-s}.$$

Thus $n'_{J_s}(t) < 0$, $\forall t > 0$ and $\forall s \in \mathbb{R} - \{0, 1\} \Rightarrow n_{J_s}(t)$ is decreasing $\forall t > 0$

As $h_{J_s}(1) = 0$, $n_{J_s}(t)$ changes the sign at $t = 1$ and therefore $h_{J_s}(t)$ changes the sign at $t = 1$. Thus for any s , we have

$$h_{J_s}(t) \begin{cases} > 0, & t < 1 \\ < 0, & t > 1 \end{cases}$$

As $\frac{p}{r} = t$, then $\frac{q}{r} = \frac{q}{p} \frac{p}{r} = \beta t$, where $\beta = \frac{q}{p}$. Therefore,

$$\sqrt{r} \frac{dJ_s}{dr} = h_{J_s}(t) + h_{J_s}(t\beta).$$

Now,

- for $t < \frac{1}{\beta}$: $h_{J_s}(t)$ and $h_{J_s}(\beta t)$ have the same sign +
- for $t > 1$: $h_{J_s}(t)$ and $h_{J_s}(\beta t)$ have the same sign -
- for $t \in \left(\frac{1}{\beta}, 1\right) \Rightarrow t\beta > 1 \Rightarrow h_{J_s}(t\beta) < 0$

- for $t \in \left(\frac{1}{\beta}, 1\right) \Rightarrow h_{J_s}(t) > 0$

Finally, for $t \in \left(\frac{1}{\beta}, 1\right)$, $h_{J_s}(t) > 0$ e $h_{J_s}(t\beta) < 0$. Thus we observe that the sign of $h_{J_s}(t) + h_{J_s}(t\beta)$ may change in $\left(\frac{1}{\beta}, 1\right)$. Now, we shall show that this happens only once.

Since h'_{J_s} is monotonically decreasing this implies that $h'_{J_s} < 0$ and we know that the function h_{J_s} changes the sign only once. This gives

$$\frac{d}{dt}(h_{J_s}(t) + h_{J_s}(t\beta)) = h'_{J_s}(t) + h'_{J_s}(t\beta) < 0.$$

Repeating the same process by substituting $t := \frac{q}{r}$ and $\frac{p}{r} = \beta t$ with $\beta = \frac{p}{q}$ we conclude that the function T'_{pq} also changes the sign at $r = q$.

(ii) For $s = 0, 1$, the result follows by the continuity of the function $\xi_s(p, q)$ with respect to s . \square

3 Asymptotic Approximation

In this section we shall bring asymptotic approximation of the measures given by (1.1) and (1.2). For this, first we shall give a general result for Csiszár's f -divergence then the other cases become as particular.

Given a function $f : (0, \infty) \rightarrow \mathbb{R}$, the f -divergence measure introduced by Csiszár's [2] is given by

$$C_f(P||Q) = \sum_{i=1}^n q_i f\left(\frac{p_i}{q_i}\right), \quad (3.1)$$

for all $P, Q \in \Gamma_n$.

The following result is well known in the literature [2].

Result 3.1. *If the function f is convex and normalized, i.e., $f(1) = 0$, then the f -divergence, $C_f(P||Q)$ is nonnegative and convex in the pair of probability distribution $(P, Q) \in \Gamma_n \times \Gamma_n$.*

Based on Result 3.1, we can prove some properties of the measures (1.1) and (1.2).

Theorem 3.1. *If f is twice differentiable at $x = 1$ and $f''(1) > 0$. Also $f(1) = 0$, then*

$$C_f(P||Q) \approx \frac{f''(1)}{2} \chi^2(P||Q). \quad (3.2)$$

Equivalently,

$$\frac{C_f(P||Q)}{\chi^2(P||Q)} \rightarrow \frac{f''(1)}{2} \text{ as } P \rightarrow Q.$$

Proof. From Taylor's series expansion, we have

$$f(x) = f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2 + k(x)(x-1)^2,$$

where $f(1) = 0$ and $k(x) \rightarrow 0$ as $x \rightarrow 1$. Hence

$$q_i f\left(\frac{p_i}{q_i}\right) = f'(1)(p_i - q_i) + \frac{f''(1)}{2} \frac{(p_i - q_i)^2}{q_i} + k\left(\frac{p_i}{q_i}\right) \frac{(p_i - q_i)^2}{q_i}.$$

Approximating $p_i \rightarrow q_i$ and summing over all $i = 1, 2, \dots, n$ we get the required result. \square

Proposition 3.1. *The following results hold:*

$$(i) \quad \zeta_s(P||Q) \approx \frac{1}{8}\chi^2(P||Q), \quad \forall s \in \mathbb{R}.$$

$$(ii) \quad \xi_s(P||Q) \approx \chi^2(P||Q), \quad \forall s \in \mathbb{R}.$$

Proof. (i) For all $x > 0$ and $s \in (-\infty, \infty)$, let us consider

$$\psi_s(x) = \begin{cases} [s(s-1)]^{-1} \left[\left(\frac{x^{1-s}+1}{2} \right) \left(\frac{x+1}{2} \right)^s - \left(\frac{x+1}{2} \right) \right], & s \neq 0, 1 \\ \frac{x}{2} \ln x - \left(\frac{x+1}{2} \right) \ln \left(\frac{x+1}{2} \right), & s = 0 \\ \left(\frac{x+1}{2} \right) \ln \left(\frac{x+1}{2\sqrt{x}} \right), & s = 1 \end{cases}, \quad (3.3)$$

in (3.1), then we have $C_f(P||Q) = \zeta_s(P||Q)$, where $\zeta_s(P||Q)$ is as given by (1.1).

We have

$$\psi'_s(x) = \begin{cases} (s-1)^{-1} \left[\frac{1}{s} \left[\left(\frac{x+1}{2x} \right)^s - 1 \right] - \frac{x^{-s}-1}{4} \left(\frac{x+1}{2} \right)^{s-1} \right], & s \neq 0, 1 \\ -\frac{1}{2} \ln \left(\frac{x+1}{2x} \right), & s = 0 \\ 1 - x^{-1} - \ln x - 2 \ln \left(\frac{2}{x+1} \right), & s = 1 \end{cases}$$

and

$$\psi''_s(x) = \left(\frac{x^{-s-1}+1}{8} \right) \left(\frac{x+1}{2} \right)^{s-2}. \quad (3.4)$$

This gives

$$\psi''_s(1) = \frac{1}{4}. \quad (3.5)$$

Expression (3.2) together with (3.4) and (3.5) completes the proof of part (i).

In particular, when $s = 0$ in (3.3) the result is obtained in Endres and Schindelin [3].

(ii) For all $x > 0$ and $s \in (-\infty, \infty)$, let us consider

$$\phi_s(x) = \begin{cases} [s(s-1)]^{-1} [x^s + x^{1-s} - (1+x)], & s \neq 0, 1 \\ (x-1) \ln x, & s = 0, 1 \end{cases}, \quad (3.6)$$

in (3.1), then we have $C_f(P||Q) = \xi_s(P||Q)$, where $\xi_s(P||Q)$ is given by (1.2).

We have

$$\phi'_s(x) = \begin{cases} [s(s-1)]^{-1} [s(x^{s-1} + x^{-s}) + x^{-s} - 1], & s \neq 0, 1 \\ 1 - x^{-1} + \ln x, & s = 0, 1 \end{cases},$$

and

$$\phi''_s(x) = x^{s-2} + x^{-s-1}. \quad (3.7)$$

This gives

$$\phi''_s(1) = 2. \quad (3.8)$$

Expression (3.8) together with (3.1) and (3.2) completes the proof of part (ii). \square

References

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